Math 579 Fall 2013 Exam 9 Solutions

1. Describe a process that yields a sequence whose generating function is given by $\frac{1}{1-\frac{x}{(1-x)^2}}\left(=\frac{(1-x)^2}{x^2-3x+1}\right)$.

   This is Example 8.14 in the text. Given a list of m objects, there are m ways to pick one distinguished object. This has generating function $\frac{x}{(1-x)^2}=\sum_{m\geq 0} mx^m$. Now, we divide a list of n objects into nonempty sublists. From each sublist we choose one element to distinguish. We now use Thm 8.13.

2. Let $k\in\mathbb{N}_0$, and consider the sequence given by $a_n=\binom{n+k}{k}$. Prove that it has generating function $A(x)=\frac{1}{(1-x)^{k+1}}$.

   We use induction on k. For $k=0$ we have the geometric series $\frac{1}{1-x}=\sum_{n\geq 0} x^n$. Suppose now that $\frac{1}{(1-x)^k}=(1-x)^{-k}=\sum_{n\geq 0}\binom{n+k-1}{k-1}x^n$. We take derivatives of each side to get $(-k)(-1)(1-x)^{-k-1}=\sum_{n\geq 0}\binom{n+k-1}{k-1}nx^{n-1}$. We drop the initial zero term, then reindex to get $\frac{k}{(1-x)^{k+1}}=\sum_{n\geq 0}\binom{n+k}{k-1}(n+1)x^n$. We divide both sides by k, and note that $\binom{n+k}{k-1}\frac{n+1}{k}=\frac{(n+k)!(n+1)}{(n+1)!(k-1)!k}=\frac{(n+k)!}{n!k!}=\binom{n+k}{k}$.

3. Consider the sequence $a_n=n^2+2n+3+(-1)^n$ for $n\geq 0$. Find and simplify the generating function for this sequence.

   We use the result of problem 2. We have $2\binom{n+2}{2}=(n+2)(n+1)=n^2+3n+2$ We have $\binom{n+1}{1}=n+1$. Hence $n^2+2n+3=2\binom{n+2}{2}-\binom{n+1}{1}+2$, so $A(x)=\frac{2}{(1-x)^3}-\frac{1}{(1-x)^2}+\frac{2}{1-x}+\frac{1}{1+x}=\frac{x^3+2x^2-3x+4}{(1-x)^3(1+x)}$.

4. Use generating functions to solve the recurrence specified by $a_0=2, a_n=3a_{n-1}+2$ $(n\geq 1)$.

   Let $A(x)=\sum_{n\geq 0} a_nx^n$. We have $A(x)-3xA(x)-\frac{2}{1-x}=\sum_{n\geq 0} a_nx^n+\sum_{n\geq 0}(-3a_n)x^{n+1}+\sum_{n\geq 0}(-2)x^n=\sum_{n\geq 0} a_nx^n+\sum_{n\geq 1}(-3a_{n-1})x^n+\sum_{n\geq 0}(-2)x^n=a_0x^0+(-2)x^0+\sum_{n\geq 1}(a_n-3a_{n-1}-2)x^n=0$. Hence $(1-3x)A(x)=\frac{2}{1-x}$ so $A(x)=\frac{2}{(1-x)(1-3x)}=\frac{3}{1-3x}+\frac{-1}{1-x}$, where the last step required partial fractions. Hence $A(x)=3\sum_{n\geq 0} 3^nx^n-\sum_{n\geq 0} 1x^n=\sum_{n\geq 0}(3^{n+1}-1)x^n$, so $a_n=3^{n+1}-1$.

5. When faced with a stack of exams to grade, I give each exam in turn either a pass or a fail. At some point (perhaps before I even start), my booze runs out and I stop grading immediately. The last thing I do is choose one of the remaining exams (and there is always at least one exam left), on which I draw a smiley face. Let a_n represent the number of ways I can do this with n exams. For example, $a_0=0$ (can't draw a smiley), $a_1=1$, $a_2=4$. Find a closed form for a_n.

   Let $G(x)=\sum_{n\geq 0} 2^nx^n=\frac{1}{1-2x}$ denote the grading part, and $S(x)=\sum_{n\geq 0} nx^n=\frac{x}{(1-x)^2}$ denote the smiley part. We combine these with the product formula $A(x)=G(x)S(x)=\frac{x}{(1-2x)(1-x)^2}=\frac{2}{1-2x}+\frac{-1}{1-x}+\frac{-1}{(1-x)^2}$, where partial fractions were used in the last step. We now have $A(x)=2\sum_{n\geq 0} 2^nx^n-\sum_{n\geq 0} 1x^n-\sum_{n\geq 0}(n+1)x^n=\sum_{n\geq 0}(2^{n+1}-1-(n+1))x^n$, so $a_n=2^{n+1}-n-2$.